

ON THE MOD-GAUSSIAN CONVERGENCE OF A SUM OVER PRIMES

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ABSTRACT. We prove mod-Gaussian convergence for a Dirichlet polynomial which approximates $\operatorname{Im} \log \zeta(1/2 + it)$. This Dirichlet polynomial is long enough to deduce Selberg's central limit theorem with an explicit error term. Moreover, assuming the Riemann hypothesis, we apply the theory of the Riemann zeta-function to extend this mod-Gaussian convergence to the complex plane. From this we obtain that $\operatorname{Im} \log \zeta(1/2 + it)$ satisfies a large deviation principle on the critical line. Results about the moments of the Riemann zeta-function follow.

1. INTRODUCTION

In this paper we study the distribution of values taken by $\log \zeta(1/2 + it)$. A breakthrough was achieved by Selberg who showed that as t varies in $[T, 2T]$, the distribution of $(\operatorname{Re} \log \zeta(1/2 + it), \operatorname{Im} \log \zeta(1/2 + it))$ is approximately Gaussian, with independent components each having expectation 0 and variance $(\log \log T)/2$. More precisely, he proved a central limit theorem which is by Lévy's continuity theorem equivalent to the statement that

$$\frac{1}{T} \int_T^{2T} e^{iu \frac{\operatorname{Re} \log \zeta(1/2 + it)}{\sqrt{(\log \log T)/2}} + iv \frac{\operatorname{Im} \log \zeta(1/2 + it)}{\sqrt{(\log \log T)/2}}} dt \rightarrow e^{-u^2/2 - v^2/2}, \quad (1.1)$$

as $T \rightarrow \infty$, where u, v are real numbers. For the case of $\operatorname{Im} \log \zeta(1/2 + it)$ see [18], [19], and also the work of Ghosh [6]. The general case is contained for instance in the book of Joyner [10]. Some of Selberg's more recent results, for example about the rate of convergence, can be found in [20] and the thesis of Tsang [22]. Initially, Selberg obtained asymptotics for the joint moments from which (1.1) follows by the method of moments. A more effective approach, which we will use as well, is contained in the work of Bombieri and Hejhal [2]. A central limit theorem for the sum over primes $(1/\sqrt{(\log \log x)/2}) \sum_{p \leq x} p^{-1/2 - iU_T}$, U_T being random variables uniformly distributed on $[T, 2T]$, $\log x = \log T/(\log \log T)^{1/4}$, follows from the mean value theorem of Montgomery and Vaughan and the method of moments. To complete the proof, they showed (see [2, Lemma 3 and Corollary]) that the L^1 -norm of $\log \zeta(1/2 + iU_T) - \sum_{p \leq x} p^{-1/2 - iU_T}$ is sufficiently small.

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The convergence in (1.1) is also a consequence of a conjecture on the behaviour of the moments of the Riemann zeta-function on the critical line (see, e.g., the work of Keating and Snaith [11] and the references therein). In the formulation of [12, Conjecture 9] it asserts that

$$e^{(u^2+v^2)(\log \log T)/4} \frac{1}{T} \int_T^{2T} e^{iu \operatorname{Re} \log \zeta(1/2+it) + iv \operatorname{Im} \log \zeta(1/2+it)} dt \rightarrow \Phi_g(u, v) \Phi_a(u, v) \quad \text{as } T \rightarrow \infty \quad (1.2)$$

locally uniformly for $u, v \in \mathbb{C}$ with $\operatorname{Im} u < 1$ and analytic functions Φ_g, Φ_a (see also [8, Conjecture 1]). This type of convergence was introduced in [9] where it is called mod-Gaussian convergence.

A precise form of the function Φ_g was conjectured by Keating and Snaith and is based on calculations in the theory of random matrices (see [11], [12, formula (18)]). The arithmetic factor Φ_a can be explained, for instance, by a similar convergence by computing the asymptotics of the characteristic function of $\sum_{n \leq x} \Lambda(n)/(n^{1/2+iU_T} \log n)$ (see [8, Theorem 2], where x has to be $O((\log T)^{2-\epsilon})$) or of the corresponding stochastic model (replace $\{p^{iU_T}\}_{p \in \mathbb{P}}$ by an independent sequence of random variables uniformly distributed on the unit circle, see [12, Example 4]).

In this paper we further investigate the distribution of the sum over primes $\sum_{p \leq x} p^{-1/2-it}$ as t varies in $[T, 2T]$ and its consequences on the distribution of the values of the Riemann zeta-function on the critical line. Thereby we will restrict ourselves to the case of $\operatorname{Im} \log \zeta(1/2 + it)$. Note that some of the arguments cannot be carried over to the case of $\operatorname{Re} \log \zeta(1/2 + it)$. It is our first aim to establish mod-Gaussian convergence if x fulfills certain conditions. Precisely, in Section 4 we prove the following:

Theorem 1.1. *Let $x = e^{\log T/N}$ and N such that x and $N/\log \log T \rightarrow \infty$ as $T \rightarrow \infty$. Then*

$$e^{u^2(\log \log x + \gamma)/4} \frac{1}{T} \int_T^{2T} e^{iu \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}}} dt \rightarrow \Phi(u) \quad \text{as } T \rightarrow \infty \quad (1.3)$$

locally uniformly for $u \in \mathbb{R}$. Here γ denotes Euler's constant and Φ is the analytic function given by

$$\Phi(u) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p}\right)^{-u^2/4} J_0\left(\frac{u}{\sqrt{p}}\right) \quad (1.4)$$

where J_0 denotes the zeroth Bessel function (see, e.g., Section 3).

One interesting point of the result seems to be the size of x . It can be chosen large enough to obtain Selberg's central limit theorem with Selberg's explicit error term (see [20, Theorem 2] and Appendix A). Moreover, we obtain the following improvement of (1.1):

Corollary 1.1. *Assume RH. Then for T sufficiently large,*

$$\frac{1}{T} \int_T^{2T} e^{iu \frac{\operatorname{Im} \log \zeta(1/2+it)}{\sqrt{(\log \log T)/2}}} dt = e^{-u^2/2} + u^2 O\left(\frac{\log \log \log T}{\log \log T}\right) + O(1/\log T)$$

uniformly for $|u| \leq \sqrt{\log \log T / \log \log \log T}$.

In Section 5 we deal with the question if the convergence in Theorem 1.1 can be extended to the complex plane. Assuming the Riemann hypothesis, we prove such a result for a weighted sum over primes.

Theorem 1.2. *Assume RH. Let $x = e^{\log T/N}$ and N such that x and $N/\log \log T \rightarrow \infty$ as $T \rightarrow \infty$. Furthermore, let f be the function $f(u) = (\pi u/2) \cot(\pi u/2)$ and $\gamma_f = -0.1080\dots$ be the constant defined by $\prod_{p \leq x} (1 - f^2(\log p / \log x)/p) = (e^{-\gamma_f} / \log x)(1 + o(1))$. Then*

$$e^{-z^2(\log \log x + \gamma_f)/4} \frac{1}{T} \int_T^{2T} e^{z \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} f\left(\frac{\log p}{\log x}\right)} dt \rightarrow \Phi(-iz) \quad \text{as } T \rightarrow \infty$$

locally uniformly for $z \in \mathbb{C}$, where Φ is given by (1.4).

More general sums are possible as well (see [7, Lemma 1] and [2, Lemma 1]). For the evaluation of γ_f see [7, proof of Lemma 6].

The crucial step from Theorem 1.1 to Theorem 1.2 is an estimate of the exponential moments of the above sum. For this purpose let $x \leq T^2$ and $h \in \mathbb{R}$. Assuming the Riemann hypothesis, we then show that there exist constants C, C' and C'' such that

$$\frac{1}{T} \int_T^{2T} e^{h \sum_{n \leq x} \frac{\Lambda(n)}{\log n} \frac{\sin(t \log n)}{\sqrt{n}} f\left(\frac{\log n}{\log x}\right)} dt \leq C'' e^{C|h| \frac{\log T}{\log x} + C' h^2 \log \log T}.$$

Note that this inequality, which is almost a subgaussian bound, is valid beyond the range which is contained in Theorem 1.1 and Theorem 1.2.

We turn to the applications of Theorem 1.2. As described above, Theorem 1.1 can be used to obtain results in connection with the central limit theorem. In addition, Theorem 1.2 yields large deviations results. Recently, Radziwiłł [16] extended the range of Selberg's central limit theorem and Soundararajan [21] proved large deviation bounds for $\operatorname{Re} \log \zeta(1/2 + it)$. By Gärtner-Ellis' theorem and Theorem 1.2, one obtains a large deviation principle (see [4, chapter 1.2] or Appendix C for the definition of the large deviation principle) from which we deduce the following two Corollaries.

Corollary 1.2. *Assume RH. Let U_T be random variables uniformly distributed on $[T, 2T]$. Then the family $(1/((\log \log T)/2)) \operatorname{Im} \log \zeta(1/2 + iU_T)$ satisfies the large deviation principle with speed $1/((\log \log T)/2)$ and rate function $I(h) = h^2/2$. For instance,*

$$\begin{aligned} \frac{1}{(\log \log T)/2} \log \left(\frac{1}{T} \lambda(\{t \in [T, 2T] : \operatorname{Im} \log \zeta(1/2 + it) \geq h(\log \log T)/2\}) \right) \\ \rightarrow -h^2/2 \quad \text{as } T \rightarrow \infty \end{aligned} \quad (1.5)$$

where $h > 0$ and λ denotes the Lebesgue measure.

Corollary 1.3. *Assume RH. Let $h \in \mathbb{R}$. Then*

$$\frac{1}{(\log \log T)/2} \log \left(\frac{1}{T} \int_T^{2T} e^{h \operatorname{Im} \log \zeta(1/2+it)} dt \right) \rightarrow h^2/2 \quad \text{as } T \rightarrow \infty.$$

Only recently, Soundararajan [21, Corollary A] completed the proof of Corollary 1.3 in the case of $\operatorname{Re} \log \zeta(1/2+it)$ by proving the upper bound. The result can be stated as follows. For all $\epsilon > 0$ and all $h > 0$ we have $(\log T)^{h^2-\epsilon} \ll_{h,\epsilon} \int_T^{2T} |\zeta(1/2+it)|^{2h} dt \ll_{h,\epsilon} (\log T)^{h^2+\epsilon}$. Note that his proof of the upper bound can be carried over to the case of $\operatorname{Im} \log \zeta(1/2+it)$ and that we use a slightly weaker upper bound in the proofs of Theorem 1.2 and Corollary 1.3.

Notation. For $y \geq 2$ and a function $g : [0, 1] \rightarrow [0, 1]$, we define

$$\begin{aligned} \Sigma_{g,y}(t) &= \sum_{p \leq y} \frac{1}{p^{1/2+it}} g\left(\frac{\log p}{\log y}\right), \\ \Sigma_{g,y}^*(t) &= \sum_{n \leq y} \frac{\Lambda(n)}{\log n} \frac{1}{n^{1/2+it}} g\left(\frac{\log n}{\log y}\right), \end{aligned}$$

$$r_{g,y}(t) = \log \zeta(1/2+it) - \Sigma_{g,y}(t), \text{ and } r_{g,y}^*(t) = \log \zeta(1/2+it) - \Sigma_{g,y}^*(t).$$

2. MOMENTS OF A SUM OVER PRIMES

We start with some standard mean value calculations. Let x be a positive real number and denote by p_1, p_2, \dots, p_n the prime numbers not exceeding x . We have

$$\begin{aligned} \frac{1}{T} \int_T^{2T} \left| \sum_{p \leq x} \frac{1}{p^{1/2+it}} \right|^{2k} dt \\ = \frac{1}{T} \int_T^{2T} \left| \sum_{\lambda_1 + \dots + \lambda_n = k} \binom{k}{\lambda_1 \dots \lambda_n} (p_1^{-\lambda_1} \dots p_n^{-\lambda_n})^{1/2+it} \right|^2 dt \end{aligned} \quad (2.1)$$

and applying the mean value theorem of Montgomery and Vaughan contained in [14] this is equal to

$$\sum_{\lambda_1 + \dots + \lambda_n = k} \binom{k}{\lambda_1 \dots \lambda_n}^2 p_1^{-\lambda_1} \dots p_n^{-\lambda_n} + \theta \frac{6\pi}{T} \sum_{\lambda_1 + \dots + \lambda_n = k} \binom{k}{\lambda_1 \dots \lambda_n}^2 \quad (2.2)$$

with $|\theta| \leq 1$. The absolute value of the remainder is bounded by $(6\pi/T)k!n^k$.

Applying $\sin(t \log p) = (p^{it} - p^{-it})/2i$ and a version of the mean value theorem contained in [17, Theorem 1.4.3] or [22, Lemma 3.1], we obtain

$$\begin{aligned}
& \frac{1}{T} \int_T^{2T} \left(\sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^{2k} dt \\
&= \frac{1}{2^{2k}} \binom{2k}{k} \sum_{\lambda_1 + \dots + \lambda_n = k} \binom{k}{\lambda_1 \dots \lambda_n}^2 p_1^{-\lambda_1} \dots p_n^{-\lambda_n} \\
&+ \theta \frac{2D}{2^{2k} T} \sum_{j=0}^{2k} \binom{2k}{j} \sqrt{\sum_{\lambda_1 + \dots + \lambda_n = j} \binom{j}{\lambda_1 \dots \lambda_n}^2} \sqrt{\sum_{\lambda_1 + \dots + \lambda_n = 2k-j} \binom{2k-j}{\lambda_1 \dots \lambda_n}^2}
\end{aligned} \tag{2.3}$$

with $|\theta| \leq 1$ and D the constant in [17, Theorem 1.4.3]. The absolute value of the remainder is bounded by $(2D/T)\sqrt{n^{2k}(2k)!}$. In the same way one can show that the absolute value of the $(2k+1)$ th moment is bounded by $(2D/T)\sqrt{n^{2k+1}(2k+1)!}$.

Now let X_1, X_2, \dots be an i.i.d. sequence of random variables uniformly distributed on the unit circle. Then we have

$$\begin{aligned}
\mathbb{E} \left[\left| \sum_{i=1}^n \frac{X_i}{\sqrt{p_i}} \right|^{2k} \right] &= \mathbb{E} \left[\left| \sum_{\lambda_1 + \dots + \lambda_n = k} \binom{k}{\lambda_1 \dots \lambda_n} \left(\frac{X_1}{\sqrt{p_1}} \right)^{\lambda_1} \dots \left(\frac{X_n}{\sqrt{p_n}} \right)^{\lambda_n} \right|^2 \right] \\
&= \sum_{\lambda_1 + \dots + \lambda_n = k} \binom{k}{\lambda_1 \dots \lambda_n}^2 p_1^{-\lambda_1} \dots p_n^{-\lambda_n}
\end{aligned}$$

and

$$\mathbb{E} \left[\left(\sum_{i=1}^n \frac{\operatorname{Im} X_i}{\sqrt{p_i}} \right)^{2k} \right] = \frac{1}{2^{2k}} \binom{2k}{k} \sum_{\lambda_1 + \dots + \lambda_n = k} \binom{k}{\lambda_1 \dots \lambda_n}^2 p_1^{-\lambda_1} \dots p_n^{-\lambda_n}. \tag{2.4}$$

Usually one chooses $x = T^a$, $0 < a < 1$, which means one saves a negative power of T in the remainder of (2.2) and (2.3) if $k < 1/a$. The main term in (2.2) is then equal to $k!(\log \log T)^k + O((\log \log T)^{k-1})$, the main contribution coming from $\lambda_i \in \{0, 1\}$ for all i . From this one obtains (see, e.g., [18, Lemma 4]) that the left hand side of (2.3) is equal to $((2k)!/(k!2^k))((\log \log T)/2)^k + O((\log \log T)^{k-1})$. The main term is a Gaussian moment with variance $(\log \log T)/2$ but the remainder is quite large. Using (2.3), one can also calculate the cumulants, provided a is chosen appropriately. Up to a negative power of T , these are equal to those of the stochastic model in (2.4) and these converge (except the second one, see formula (3.3)) even without normalization.

3. BESSEL FUNCTIONS

The Bessel functions appear in the Fourier expansion of the function $e^{iu \sin \theta}$,

$$e^{iu \sin \theta} = \sum_{k=-\infty}^{\infty} J_k(u) e^{ik\theta}. \quad (3.1)$$

Explicitly the k th Bessel function $J_k(u)$ is given by

$$J_k(u) = \sum_{n=0}^{\infty} \frac{(-1)^n (u/2)^{k+2n}}{n!(k+n)!}$$

for $k \geq 0$ and given by the relation $J_k(u) = (-1)^k J_{-k}(u)$ for $k < 0$ (for these and more facts about Bessel functions see, e.g., the book of Andrews, Askey, and Roy [1]).

Now let X_1, X_2, \dots again be an i.i.d. sequence of random variables uniformly distributed on the unit circle. We have

$$\mathbb{E}[e^{iu \operatorname{Im} X_1}] = \frac{1}{2\pi} \int_0^{2\pi} e^{iu \sin \theta} d\theta = J_0(u). \quad (3.2)$$

Applying (3.2), Weierstrass' product formula, and Merten's formula, we obtain that $\sum_{i=1}^{\pi(x)} \operatorname{Im} X_i / \sqrt{p_i}$ converges mod-Gaussian (see [9]), i.e.

$$e^{u^2(\log \log x + \gamma)/4} \mathbb{E} \left[e^{iu \sum_{i=1}^{\pi(x)} \frac{\operatorname{Im} X_i}{\sqrt{p_i}}} \right] \rightarrow \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p} \right)^{-u^2/4} J_0 \left(\frac{u}{\sqrt{p}} \right) \quad (3.3)$$

as $x \rightarrow \infty$, locally uniformly for $u \in \mathbb{C}$, where $\pi(x)$ denotes the number of primes not exceeding x .

4. MOD-CONVERGENCE OF A SUM OVER PRIMES

By means of (2.3), (2.4), and the analogous results for odd integers, we can apply for fixed x the method of moments and obtain the following weak convergence

$$\frac{1}{T} \int_T^{2T} e^{iu \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}}} dt \rightarrow \prod_{p \leq x} J_0 \left(\frac{u}{\sqrt{p}} \right) \quad \text{as } T \rightarrow \infty. \quad (4.1)$$

Another proof of (4.1) is contained in [13, Theorem 5.1]. The techniques used therein can be applied to get Theorem 1.1 and Theorem 1.2 for the choice $x = (\log T)^{2-\epsilon}$, $\epsilon > 0$ arbitrary. The improvement of Theorem 1.1 follows from combining (3.3) with the following proposition.

Proposition 4.1. *Let $c > 1$ be a constant. Define $x = e^{\log T/N}$ with $N = (c'ec^2/4) \log \log T$, $c' > 1$ another constant. Then for T sufficiently large independent of c and c' ,*

$$\frac{1}{T} \int_T^{2T} e^{iu \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}}} dt = \prod_{p \leq x} J_0 \left(\frac{u}{\sqrt{p}} \right) + O((1/c')^N + (2c^2/\log x)^N) \quad (4.2)$$

uniformly for $|u| \leq c$, $u \in \mathbb{R}$. The remainder is $o(\exp(-c^2(\log \log T)/4))$, if $c' \log c' > 1/e$.

Proof. From the Taylor expansion $e^{iu} = \sum_{k \leq 2N'-1} (iu)^k/k! + \theta u^{2N'}/(2N')!$, $u \in \mathbb{R}$, with $|\theta| \leq 1$ and $N' = \lfloor N \rfloor$, we obtain

$$\begin{aligned} \frac{1}{T} \int_T^{2T} e^{iu \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}}} dt &= \sum_{k \leq 2N'-1} \frac{(iu)^k}{k!} \frac{1}{T} \int_T^{2T} \left(\sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^k dt \\ &\quad + \theta \frac{u^{2N'}}{(2N')!} \frac{1}{T} \int_T^{2T} \left(\sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^{2N'} dt \quad (4.3) \end{aligned}$$

with $|\theta| \leq 1$. The remainder is by (2.3)

$$O\left(\frac{c^{2N'}}{N'!} \frac{1}{2^{2N'}} \left(\sum_{p \leq x} \frac{1}{p}\right)^{N'} + \frac{(c^2 n)^{N'}}{T}\right).$$

Using the bound $(N')! \geq (N'/e)^{N'}$, elementary results in the theory of primes, namely the formulas $\sum_{p \leq x} 1/p = \log \log x + c_1 + O(1/\log x)$ and $n = \pi(x) \leq 2x/\log x$, and finally $N' = \lfloor N \rfloor$, this is

$$O\left(\left(\frac{ec^2 \log \log T}{4N}\right)^N + \left(\frac{2c^2}{\log x}\right)^N\right) = O\left(\left(\frac{1}{c'}\right)^N + \left(\frac{2c^2}{\log x}\right)^N\right)$$

for sufficiently large T independent of c, c' (since $c, c' > 1$).

Now let X_1, X_2, \dots be an i.i.d. sequence of random variables uniformly distributed on the unit circle. The moments in (4.3) are by (2.3) and (2.4) equal to those of the stochastic model plus a remainder which is bounded by $(2D/T)\sqrt{n^k k!}$. The resulting remainders in (4.3), $k \leq 2N' - 1$, add up to $O((c^2 n)^N/T) = O(2c^2/\log x)^N$. Hence, (4.3) is equal to

$$\sum_{k \leq 2N'-1} \frac{(iu)^k}{k!} \mathbb{E} \left(\sum_{i=1}^n \frac{\operatorname{Im} X_i}{\sqrt{p_i}} \right)^k + O((1/c')^N + (2c^2/\log x)^N)$$

for sufficiently large T . Applying the above Taylor expansion again, we obtain

$$\begin{aligned} \prod_{p \leq x} J_0\left(\frac{u}{\sqrt{p}}\right) &= \mathbb{E} \left[e^{iu \sum_{i=1}^n \frac{\operatorname{Im} X_i}{\sqrt{p_i}}} \right] \\ &= \sum_{k \leq 2N'-1} \frac{(iu)^k}{k!} \mathbb{E} \left(\sum_{i=1}^n \frac{\operatorname{Im} X_i}{\sqrt{p_i}} \right)^k + \theta \frac{u^{2N'}}{(2N')!} \mathbb{E} \left(\sum_{i=1}^n \frac{\operatorname{Im} X_i}{\sqrt{p_i}} \right)^{2N'} \end{aligned}$$

with $|\theta| \leq 1$. The remainder already appeared in (4.3) and is $O((1/c')^N)$ for sufficiently large T . This completes the proof. \square

5. MOD-CONVERGENCE IN THE COMPLEX PLANE

In this section we prove Theorem 1.2. Our initial point is an explicit formula obtained by Goldston [7, Lemma 1] assuming RH, namely

$$\begin{aligned} \operatorname{Im} \log \zeta(1/2 + it) &= - \sum_{n \leq x} \frac{\Lambda(n)}{\log n} \frac{\sin(t \log n)}{\sqrt{n}} f\left(\frac{\log n}{\log x}\right) \\ &+ \sum_{\gamma} \sin((t - \gamma) \log x) \int_0^{\infty} \frac{u}{u^2 + ((t - \gamma) \log x)^2} \frac{du}{\sinh u} + O\left(\frac{1}{t(\log x)^2}\right) \end{aligned} \quad (5.1)$$

for $4 \leq x \leq t^2$, $t \neq \gamma$, and $f(u) = (\pi u/2) \cot(\pi u/2)$. Furthermore, we will use the following result on the distribution of $\operatorname{Im} \log \zeta(1/2 + it)$ which is valid under RH: For every $h \in \mathbb{R}$ there exist constants C' and C'' such that the following subgaussian bound holds

$$\frac{1}{T} \int_T^{2T} e^{h \operatorname{Im} \log \zeta(1/2 + it)} dt \leq C'' e^{C' h^2 \log \log T}. \quad (5.2)$$

Soundararajan [21] proved such a result for $\operatorname{Re} \log \zeta(1/2 + it)$ and his arguments can be carried over to the case of $\operatorname{Im} \log \zeta(1/2 + it)$ by using [18, Theorem 1]. In fact, he obtained an upper bound of nearly the conjectured order of magnitude. We prove (compare to [2, Lemma 3 and Corollary]):

Proposition 5.1. *Assume RH. Let $2 \leq x \leq T^2$, $f(u) = (\pi u/2) \cot(\pi u/2)$, and $h \in \mathbb{R}$. Then there exist constants C, C' and C'' such that*

$$\frac{1}{T} \int_T^{2T} e^{h \sum_{n \leq x} \frac{\Lambda(n)}{\log n} \frac{\sin(t \log n)}{\sqrt{n}} f\left(\frac{\log n}{\log x}\right)} dt \leq C'' e^{C|h| \frac{\log T}{\log x} + C' h^2 \log \log T}.$$

Proof of Theorem 1.2. We assume that $|z| \leq c$, $z \in \mathbb{C}$, $c > 1$ an arbitrary constant. With the notation defined at the end of the introduction we have $-\operatorname{Im} \Sigma_{f,x}(t) = \sum_{p \leq x} (\sin(t \log p)/\sqrt{p}) f(\log p/\log x)$. Let $N' = \lfloor N/2 \rfloor$. Since $z = h + iu$ is a complex number, we have to use the Taylor expansion $e^z = \sum_{k \leq 2N'-1} z^k/k! + \theta e^h (|z|^{2N'}/(2N')!)$ with $|\theta| \leq 1$. We discuss the remainder which is the main difference to the proof of Theorem 1.1. It is equal to

$$\theta \frac{c^{2N'}}{(2N')!} \frac{1}{T} \int_T^{2T} (\operatorname{Im} \Sigma_{f,x}(t))^{2N'} e^{-h \operatorname{Im} \Sigma_{f,x}(t)} dt$$

with $|\theta| \leq 1$. Applying the Cauchy-Schwarz inequality, the absolute value can be bounded by

$$\frac{c^{2N'}}{(2N')!} \sqrt{\frac{1}{T} \int_T^{2T} (\operatorname{Im} \Sigma_{f,x}(t))^{4N'} dt} \sqrt{\frac{1}{T} \int_T^{2T} e^{-2h \operatorname{Im} \Sigma_{f,x}(t)} dt}.$$

Using $\sum_{p \neq n \leq x} (\Lambda(n)/\log n)(\sin(t \log n)/\sqrt{n})f(\log n/\log x) = O(\log \log T)$, say $\leq C'c^2N$, and Proposition 5.1, this is

$$\begin{aligned} & O \left(\frac{c^{2N'}}{(2N')!} \sqrt{\frac{(4N')!}{(2N')!2^{4N'}} (\log \log T)^{2N'} + \frac{\sqrt{(4N')!n^{4N'}}}{T} \sqrt{e^{2C_cN+5C'c^2N}}} \right) \\ & = O \left(\left(\frac{ec^2 \log \log T e^{2C_c+5C'c^2}}{N} \right)^{N/2} + \sqrt{\frac{(c^2ne^{2C_c+5C'c^2})^N}{T}} \right) \end{aligned} \quad (5.3)$$

for sufficiently large T . Since $N/\log \log T$ and $\log x$ tend to infinity these terms times $\exp(c^2(\log \log x)/4)$ converge to zero. The remaining steps are similar to those in the proof of Proposition 4.1 and in Section 3. \square

Proof of Proposition 5.1. From the formulas (5.1), (5.2) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} e^{h \sum_{n \leq x} \frac{\Lambda(n)}{\log n} \frac{\sin(t \log n)}{\sqrt{n}} f\left(\frac{\log n}{\log x}\right)} dt \\ & \leq C''' e^{2C'h^2 \log \log T} \sqrt{\frac{1}{T} \int_T^{2T} e^{2h \sum_{\gamma} \sin((t-\gamma) \log x) \int_0^\infty \frac{u}{u^2 + ((t-\gamma) \log x)^2} \frac{du}{\sinh u}} dt} \end{aligned}$$

where C''' is a constant. The absolute value of the sum over zeros is bounded by a constant times

$$\sum_{|(t-\gamma) \log x| \leq 1} 1 + \sum_{|(t-\gamma) \log x| > 1} \frac{1}{((t-\gamma) \log x)^2} \quad (5.4)$$

and therefore it suffices to deal with the exponential moments of (5.4) with $h \geq 0$. Using the Cauchy-Schwarz inequality again, we obtain

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} e^{h \sum_{|(t-\gamma) \log x| \leq 1} 1 + h \sum_{|(t-\gamma) \log x| > 1} \frac{1}{((t-\gamma) \log x)^2}} dt \\ & \leq \sqrt{\frac{1}{T} \int_T^{2T} e^{2h \sum_{|(t-\gamma) \log x| \leq 1} 1} dt} \sqrt{\frac{1}{T} \int_T^{2T} e^{2h \sum_{|(t-\gamma) \log x| > 1} \frac{1}{((t-\gamma) \log x)^2}} dt}. \end{aligned}$$

We start with the first term, using the following fact on the number of zeros (see [3, (1) of Ch. 15])

$$N(t) = \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O(1/t), \quad (5.5)$$

where $t \neq \gamma$ and $S(t) = (1/\pi) \operatorname{Im} \log \zeta(1/2 + it)$. We compute (note that $h \geq 0$)

$$\begin{aligned}
& \frac{1}{T} \int_T^{2T} e^{h \sum_{|(t-\gamma) \log x| \leq 1} 1} dt \\
&= \frac{1}{T} \int_T^{2T} e^{h \left(N\left(t + \frac{1}{\log x}\right) - N\left(t - \frac{1}{\log x}\right) \right)} dt \\
&\leq \frac{1}{T} \int_T^{2T} e^{Ch \frac{\log t}{\log x} + h \left(S\left(t + \frac{1}{\log x}\right) - S\left(t - \frac{1}{\log x}\right) \right)} dt \\
&\leq e^{Ch \frac{\log T}{\log x}} \sqrt{\frac{1}{T} \int_T^{2T} e^{2hS\left(t + \frac{1}{\log x}\right)} dt} \sqrt{\frac{1}{T} \int_T^{2T} e^{-2hS\left(t - \frac{1}{\log x}\right)} dt} \\
&\leq C'' e^{Ch \frac{\log T}{\log x} + 4C'(h/\pi)^2 \log \log T}.
\end{aligned}$$

In the last step we used (5.2). Next, we divide the sum over $|(t-\gamma) \log x| > 1$ into $|t-\gamma| \geq T$, $1 < |t-\gamma| < T$, and $1/\log x < |t-\gamma| \leq 1$.

For $t \in [T, 2T]$ we have

$$\sum_{|t-\gamma| \geq T} \frac{1}{((t-\gamma) \log x)^2} = O\left(\sum_{\gamma} \frac{1}{\gamma^2 (\log x)^2}\right) = O\left(\frac{1}{(\log x)^2}\right).$$

For the second sum we use the fact that $N(t+1) - N(t) = O(1 + \log^+ |t|)$ (see [3, (2) of Ch. 15]). For $t \in [T, 2T]$ we obtain

$$\begin{aligned}
& \sum_{1 < |t-\gamma| < T} \frac{1}{((t-\gamma) \log x)^2} \\
&\leq \sum_{k=1}^{[T]-1} \frac{N(t+k+1) - N(t+k)}{k^2 (\log x)^2} + \sum_{k=1}^{[T]-1} \frac{N(t-k) - N(t-k-1)}{k^2 (\log x)^2} \\
&= O\left(\sum_{k=1}^{[T]-1} \frac{\log T}{k^2 (\log x)^2}\right) = O\left(\frac{\log T}{(\log x)^2}\right).
\end{aligned}$$

Next, we consider the sum over $1/\log x < \gamma - t \leq 1$. We have

$$\sum_{1/\log x < \gamma - t \leq 1} \frac{1}{((t-\gamma) \log x)^2} \leq \sum_{j=1}^M \frac{N\left(t + \frac{k_j}{\log x}\right) - N\left(t + \frac{k_{j-1}}{\log x}\right)}{k_{j-1}^2} \quad (5.6)$$

where $1 = k_0 < k_1 < \dots < k_M = \log x$. By (5.5), this is bounded by, recall $t \in [T, 2T]$,

$$\sum_{j=1}^M \left(\frac{C(k_j - k_{j-1}) \log T}{(\log x) k_{j-1}^2} + \frac{S\left(t + \frac{k_j}{\log x}\right) - S\left(t + \frac{k_{j-1}}{\log x}\right)}{k_{j-1}^2} \right).$$

We choose $k_j = 2^{j/2}$ and bound the left hand side of (5.6) by

$$\sqrt{2}C \frac{\log T}{\log x} + \sum_{j=1}^M \frac{S\left(t + \frac{2^{j/2}}{\log x}\right) - S\left(t + \frac{2^{(j-1)/2}}{\log x}\right)}{2^{j-1}}.$$

Using $\mathbb{E}e^{h \sum_{j=1}^M X_j/2^j} \leq (\prod_{j=1}^{M-1} (\mathbb{E}e^{hX_j})^{1/2^j}) (\mathbb{E}e^{(h/2)X_M})^{1/2^{M-1}}$, which follows from repeated application of the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} e^{h \sum_{1/\log x < \gamma-t \leq 1} \frac{1}{((t-\gamma)\log x)^2}} dt \\ & \leq e^{\sqrt{2}Ch \frac{\log T}{\log x}} \frac{1}{T} \int_T^{2T} e^{h \sum_{j=1}^M \frac{1}{2^{j-1}} \left(S\left(t + \frac{2^{j/2}}{\log x}\right) - S\left(t + \frac{2^{(j-1)/2}}{\log x}\right) \right)} dt \\ & \leq e^{\sqrt{2}Ch \frac{\log T}{\log x}} C''' e^{16C'(h/\pi)^2 \log \log T}. \end{aligned}$$

The same bound is true for the sum over $1/\log x < t - \gamma \leq 1$. The claim now follows from putting together all these estimates. \square

6. PROOF OF COROLLARY 1.1

In this section we prove a second order expansion of $\text{Im} \log \zeta(1/2 + it)$ from which Corollary 1.1 easily follows.

Proposition 6.1. *Assume RH. Let $c > 1$ be a constant. Define $x = e^{\log T/N}$ with $N = (c'ec^2/4) \log \log T$, $c' > 4$ another constant. Then for T sufficiently large,*

$$\begin{aligned} \frac{1}{T} \int_T^{2T} e^{iu \text{Im} \log \zeta(1/2+it)} dt &= \prod_{p \leq x} J_0\left(\frac{u}{\sqrt{p}}\right) + u^2 O(\log \log \log T) \\ &\quad + O(N(c'/4)^{-N/2}) \quad (6.1) \end{aligned}$$

uniformly for $|u| \leq c$, $u \in \mathbb{R}$.

Proof. From $\text{Im} \log \zeta(1/2+it) = \text{Im} \Sigma_{1,x}(t) + \text{Im} r_{1,x}(t)$ and Taylor's theorem, we obtain

$$\begin{aligned} \frac{1}{T} \int_T^{2T} e^{iu \text{Im} \log \zeta(1/2+it)} dt &= \frac{1}{T} \int_T^{2T} e^{iu \text{Im} \Sigma_{1,x}(t)} dt \\ &\quad + iu \frac{1}{T} \int_T^{2T} \text{Im} r_{1,x}(t) e^{iu \text{Im} \Sigma_{1,x}(t)} dt + \theta \frac{u^2}{2} \frac{1}{T} \int_T^{2T} (\text{Im} r_{1,x}(t))^2 dt \end{aligned}$$

with $|\theta| \leq 1$. By Proposition 4.1, the first term is equal to $\prod_{p \leq x} J_0(u/\sqrt{p})$ plus $O(c'^{-N})$ for sufficiently large T . The third term is equal to $u^2 O(\log \log \log T)$ by [22, Corollary of Theorem 5.1]. To compute the second term, we show

that

$$\begin{aligned} \frac{i}{T} \int_T^{2T} \operatorname{Im} \Sigma_{1,x}(t) e^{iu \operatorname{Im} \Sigma_{1,x}(t)} dt &= \sum_{p \leq x} \frac{-1}{\sqrt{p}} J_1\left(\frac{u}{\sqrt{p}}\right) \prod_{\substack{q \leq x \\ q \neq p}} J_0\left(\frac{u}{\sqrt{q}}\right) \\ &\quad + O(Nc'^{-N}) \end{aligned} \quad (6.2)$$

and

$$\begin{aligned} \frac{i}{T} \int_T^{2T} \operatorname{Im} \log \zeta(1/2 + it) e^{iu \operatorname{Im} \Sigma_{1,x}(t)} dt &= \sum_{\substack{p \leq x \\ k \text{ odd}}} \frac{-1}{k\sqrt{p}^k} J_k\left(\frac{u}{\sqrt{p}}\right) \prod_{\substack{q \leq x \\ q \neq p}} J_0\left(\frac{u}{\sqrt{q}}\right) \\ &\quad + O(N(c'/4)^{-N/2}) \end{aligned} \quad (6.3)$$

uniformly in $|u| \leq c$, for sufficiently large T . The sum of (6.2) and (6.3) is contained in the remainders of (6.1) (by applying, for instance, the inequality $|J_k(u)| \leq (|u|/2)^k/k!$, $u \in \mathbb{R}$).

We only prove (6.3), the proof of (6.2) is similar to the proof of Proposition 4.1. Let $N' = \lfloor N/2 \rfloor$. By Taylor's theorem, the left hand side is equal to

$$\begin{aligned} \sum_{k \leq 2N'-1} \frac{(iu)^k}{k!} \frac{i}{T} \int_T^{2T} \operatorname{Im} \log \zeta(1/2 + it) (\operatorname{Im} \Sigma_{1,x}(t))^k dt \\ + \theta \frac{u^{2N'}}{(2N')!} \frac{1}{T} \int_T^{2T} |\operatorname{Im} \log \zeta(1/2 + it)| (\operatorname{Im} \Sigma_{1,x}(t))^{2N'} dt \end{aligned}$$

with $|\theta| \leq 1$. After applying the Cauchy-Schwarz inequality, the remainder is by (5.3) and [19, Theorem 6] equal to $O((\log \log T)^{1/2} (c'/4)^{-N/2})$ for sufficiently large T . The remaining moments can be computed by using the following lemma which is a modification of [18, Lemma 5] and [7, equation (6.3)] and serves as a substitute for the mean value theorem of Montgomery and Vaughan in Section 2.

Lemma. *Assume RH. Let $k, h \leq T$ be two positive integers with $(k, h) = 1$. Then*

$$\begin{aligned} \int_T^{2T} \log \zeta(1/2 + it) \left(\frac{k}{h}\right)^{it} dt &= \frac{T\Lambda(k)}{\sqrt{k} \log k} + O(\sqrt{kh} \log T), \quad h = 1 \\ &\quad O(\sqrt{kh} \log T), \quad h \neq 1, \\ \int_T^{2T} \operatorname{Im} \log \zeta(1/2 + it) \left(\frac{k}{h}\right)^{it} dt &= \frac{-iT\Lambda(k)}{2\sqrt{k} \log k} + O(\sqrt{kh} \log T), \quad h = 1 \\ &= \frac{iT\Lambda(h)}{2\sqrt{h} \log h} + O(\sqrt{kh} \log T), \quad k = 1 \\ &\quad O(\sqrt{kh} \log T), \quad h, k \neq 1. \end{aligned} \quad (6.4)$$

We denote by p_1, p_2, \dots, p_n the prime numbers not exceeding x . Let X_1, X_2, \dots be an i.i.d. sequence of random variables uniformly distributed on the unit circle. Furthermore, let $k, h \leq T$ be positive integers with $k/h = p_1^{k_1} \cdots p_n^{k_n}$. Then (6.4) can be written as

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \operatorname{Im} \log \zeta(1/2 + it) (p_1^{k_1} \cdots p_n^{k_n})^{it} dt \\ &= \mathbb{E} \left[- \sum_{i=1}^n \operatorname{Im} \log(1 - \overline{X}_i / \sqrt{p_i}) X_1^{k_1} \cdots X_n^{k_n} \right] + O\left(\frac{1}{T} \sqrt{p_1^{|k_1|} \cdots p_n^{|k_n|}} \log T\right) \end{aligned}$$

from which we conclude similarly as in Section 2 that

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \operatorname{Im} \log \zeta(1/2 + it) (\operatorname{Im} \Sigma_{1,x}(t))^k dt \\ &= \mathbb{E} \left[- \sum_{i=1}^n \operatorname{Im} \log(1 - \overline{X}_i / \sqrt{p_i}) \left(\sum_{i=1}^n \frac{\operatorname{Im} \overline{X}_i}{\sqrt{p_i}} \right)^k \right] + O\left(\frac{n^k \log T}{T}\right). \end{aligned}$$

(6.3) now follows by proceeding as in the proof of Proposition 4.1 by using

$$\begin{aligned} & \mathbb{E} \left[- \sum_{i=1}^n \operatorname{Im} \log(1 - \overline{X}_i / \sqrt{p_i}) e^{iu \sum_{i=1}^n \frac{\operatorname{Im} \overline{X}_i}{\sqrt{p_i}}} \right] \\ &= \frac{1}{i} \sum_{\substack{p \leq x \\ k \geq 1 \text{ odd}}} \frac{-1}{k \sqrt{p^k}} J_k\left(\frac{u}{\sqrt{p}}\right) \prod_{\substack{q \leq x \\ q \neq p}} J_0\left(\frac{u}{\sqrt{q}}\right) \end{aligned}$$

and a Taylor expansion of the left hand side. Note that the above formula follows from plugging in (3.1). \square

7. PROOF OF COROLLARY 1.2 AND 1.3

Proof of Corollary 1.2. Let $x \geq 2$ be as in Theorem 1.2 with the additional property that $N = O(\log \log T)$. Furthermore, let $V = (\log \log T)/M$. We consider the cases that M is a constant or that M is equal to $\log \log \log T$ times a constant. In both cases we have $V \leq N$ for sufficiently large T . We start showing that then there exists a constant $c > 0$ such that

$$(1/T) \lambda(\{t \in [T, 2T] : |\operatorname{Im} r_{f,x}(t)| \geq cV\}) \leq e^{-(1-o(1))V \log V} \quad (7.1)$$

for sufficiently large T such that $V \leq N$.

To prove (7.1), we decompose

$$\operatorname{Im} r_{f,x} = \operatorname{Im}(r_{g,T^{1/V}}^* + (\Sigma_{g,T^{1/V}}^* - \Sigma_{g,T^{1/V}}) + (\Sigma_{g,T^{1/V}} - \Sigma_{g,x}) + \Sigma_{g-f,x}).$$

If $|\operatorname{Im} r_{f,x}(t)| \geq cV$, there exists a summand on the right hand side greater or equal to $cV/4$ in absolute value. Applying the union bound, we have to deal with four terms. Three of them already contain only Dirichlet polynomials. If we choose Selberg's function $g(u) = e^{-2u} \min(1, 2(1-u))$, we can apply

[18, Theorem 1] (which states that for $2 \leq y \leq t^2$, assuming RH, $\text{Im } r_{g,y}^*(t) = O(|\Sigma_{\tilde{g},y}(t)|) + O(\log t / \log y)$, where $\tilde{g}(u) = ug(u)$) to the remaining term. For c large enough (to bound the remainder in Selberg's formula by $cV/8$), we obtain

$$(1/T)\lambda(\{t \in [T, 2T] : |\text{Im } r_{g,T^{1/V}}^*(t)| \geq cV/4\}) \leq \\ (1/T)\lambda\left(\left\{t \in [T, 2T] : \left|\frac{c'}{\log T^{1/V}} \sum_{n \leq T^{1/V}} \frac{\Lambda(n)}{n^{1/2+it}} g\left(\frac{\log n}{\log T^{1/V}}\right)\right| \geq cV/8\right\}\right)$$

where c' is another constant. Now we can apply the mean value estimates described in Appendix B and Markov's inequality to bound these four terms by

$$\frac{(Ak)^k}{V^{2k}} (\log M + \log(N/\log \log T) + O(1))^k$$

where k is an integer which has to satisfy $k \leq V \leq N$. Hence, if we choose $k = \lfloor V \rfloor$, (7.1) follows.

Formula (1.5) follows from the large deviation principle stated in the Corollary. Before we prove this large deviation principle, we give a proof of (1.5) which yields explicit upper and lower bounds. In [15] it is shown that Theorem 1.2 yields the following exact large deviations result

$$(1/T)\lambda(\{t \in [T, 2T] : \text{Im } \Sigma_{f,x}(t) \geq h(\log \log x + \gamma_f)/2\}) \\ = \Phi(ih) \frac{e^{-(\log \log x + \gamma_f)h^2/4}}{h\sqrt{\pi \log \log x}} (1 + o(1)), \quad (7.2)$$

as $T \rightarrow \infty$, where $h > 0$ may vary in a compact set. The proof therein is based on the lemma in Appendix A and an exponential change of measure, a method which traces back to Cramér (see [4, Theorem 3.7.4]). Note that Radziwiłł [16] proved an analogous formula for the choice $N = (\log \log T)^2$ without assuming RH.

From $\text{Im } \log \zeta(1/2 + it) = \text{Im } \Sigma_{f,x}(t) + \text{Im } r_{f,x}(t)$, we obtain

$$\lambda(\{t \in [T, 2T] : \text{Im } \log \zeta(1/2 + it) \geq h(\log \log T)/2\})/T \\ \leq \lambda(\{t \in [T, 2T] : \text{Im } r_{f,x}(t) \geq cV\})/T \\ + \lambda(\{t \in [T, 2T] : \text{Im } \Sigma_{f,x}(t) \geq (h - 2c/M)(\log \log T)/2\})/T$$

and

$$\lambda(\{t \in [T, 2T] : \text{Im } \log \zeta(1/2 + it) \geq h(\log \log T)/2\})/T \\ \geq -\lambda(\{t \in [T, 2T] : \text{Im } r_{f,x}(t) \leq -cV\})/T \\ + \lambda(\{t \in [T, 2T] : \text{Im } \Sigma_{f,x}(t) \geq (h + 2c/M)(\log \log T)/2\})/T.$$

If M is chosen such that $M \rightarrow \infty$, the last terms in the above inequalities are, by (7.2), equal to

$$e^{-h^2/2(\log \log x)/2(1+o(1))}.$$

After taking the logarithm and dividing by $(\log \log T)/2$, this term converges to $-h^2/2$. The formula would follow from the above inequalities if we can prove a bound $(1/T)\lambda(\{t \in [T, 2T] : |\operatorname{Im} r_{f,x}(t)| \geq cV\}) \leq \exp(-c'' \log \log T)$ with constants $c > 0$ and $c'' > h^2/4$, for sufficiently large T . But this follows from (7.1) if we choose M equal to $\log \log \log T$ times a sufficiently small constant.

Finally, we turn to the general proof of the large deviation principle. By Theorem 1.2, Theorem C.1 (see [4, Theorem 2.3.6]), and the fact that $\log \log x / \log \log T \rightarrow 1$, the family $(1/((\log \log T)/2)) \operatorname{Im} \Sigma_{f,x}(U_T)$ satisfies the large deviation principle with speed $1/((\log \log T)/2)$ and rate function $I(h) = h^2/2$. By (7.1), we obtain

$$\begin{aligned} \frac{1}{(\log \log T)/2} \log \left(\frac{1}{T} \lambda(\{t \in [T, 2T] : |\operatorname{Im} r_{f,x}(t)| \geq c\delta \log \log T\}) \right) \\ \leq -2\delta(1 - o(1))(\log \log \log T + \log \delta) \end{aligned}$$

for all $\delta > 0$ and sufficiently large T . As $T \rightarrow \infty$, the right hand side goes to $-\infty$. Hence, by definition, $(1/((\log \log T)/2)) \operatorname{Im} \log \zeta(1/2 + iU_T)$ and $(1/((\log \log T)/2)) \Sigma_{f,x}(U_T)$ are exponentially equivalent (see [4, Definition 4.2.10]). We now apply [4, Theorem 4.2.13], which states that if two families of random variables are exponentially equivalent, and one of them satisfies the large deviation principle with good rate function I , then the same large deviation principle holds for the other family. Hence, Corollary 1.2 is proved. \square

Proof of Corollary 1.3. The asserted formula is exactly content of Varadhan's integral lemma (see Theorem C.2 or [4, Theorem 4.3.1]). The assumptions of the theorem are satisfied by Corollary 1.2 and equation (5.2) \square

APPENDIX A. A RESULT OF SELBERG

In this appendix we briefly discuss Selberg's result about the rate of convergence in the central limit theorem of $\operatorname{Im} \log \zeta(1/2 + it)$ (see [20, Theorem 2] and [22, Theorem 6.2]). From Theorem 1.1 we obtain

Lemma. *Let $x = e^{\log T/N}$ and N such that x and $N/\log \log T \rightarrow \infty$ as $T \rightarrow \infty$. Assume further that $N = O(\log \log T)$. Then*

$$\begin{aligned} \sup_{a < b} \left(\frac{1}{T} \lambda \left(\left\{ t \in [T, 2T] : \frac{1}{\sqrt{(\log \log x + \gamma)/2}} \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \in [a, b] \right\} \right) \right. \\ \left. - \int_a^b e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} \right) = O(1/\sqrt{\log \log T}). \end{aligned}$$

In fact, the right hand side can be replaced by $o(1/\sqrt{\log \log T})$ (see [5, Theorem XVI.4.1]).

Proof. We denote by $\phi_n(u)$ the left hand side of (1.3). Using [5, XVI.3, formula 3.13] we can bound the right hand side by $(\log \log x / \log \log T \rightarrow 1)$

$$\frac{2}{\pi} \int_{-c\sqrt{\log \log x}}^{c\sqrt{\log \log x}} e^{-u^2/2} |(\phi_n(u/\sqrt{(\log \log x + \gamma)/2}) - 1)/u| du + O\left(\frac{1}{c\sqrt{\log \log T}}\right).$$

We choose $c > 0$ such that $J_0(u)$ has no zeros for $|u| \leq c$. An inspection of the proof of Proposition 4.1 shows that $\phi_n(u) = \phi(u)(1 + O(|u|/\log x))$, $|u| \leq c$. On the other hand, we have $\phi(u/\sqrt{(\log \log x + \gamma)/2}) = 1 + O(|u|/\sqrt{\log \log T})$, $|u| \leq c\sqrt{\log \log x}$. Plugging in these estimates gives the lemma. \square

This lemma combined with the bound (see [22, Lemma 6.3])

$$|\{t \in [T, 2T] : |r_{1,x}(t)| \geq c \log \log \log T\}| = O(1/\sqrt{\log \log T})$$

where $c > 0$ is a constant yields Selberg's result

$$\sup_{a < b} \left(\frac{1}{T} \lambda \left(\left\{ t \in [T, 2T] : \frac{\operatorname{Im} \log \zeta(1/2 + it)}{\sqrt{(\log \log T)/2}} \in [a, b] \right\} \right) - \int_a^b e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} \right) = O\left(\frac{\log \log \log T}{\sqrt{\log \log T}}\right).$$

APPENDIX B. MEAN VALUE ESTIMATES

For completeness we present some standard mean value estimates which we applied in the proof of Corollary 1.2. For this purpose let x and y be positive real numbers, a_p and b_p be complex numbers with $|a_p| \leq 1$ and $|b_p| \leq \log p / \log x$, and k be a positive integer. By repeating the arguments in (2.1) and (2.2), we obtain

$$\begin{aligned} \frac{1}{T} \int_T^{2T} \left| \sum_{p \leq x} \frac{a_p}{p^{1+2it}} \right|^{2k} dt &\leq k! \left(\sum_{p \leq x} \frac{1}{p^2} \right)^k + 6\pi k! (\pi(x))^k / T, \\ \frac{1}{T} \int_T^{2T} \left| \sum_{p \leq x} \frac{b_p}{p^{1/2+it}} \right|^{2k} dt &\leq k! \frac{1}{(\log x)^k} \left(\sum_{p \leq x} \frac{\log p}{p} \right)^k + 6\pi k! (\pi(x))^k / T, \\ \frac{1}{T} \int_T^{2T} \left| \sum_{y < p \leq x} \frac{a_p}{p^{1/2+it}} \right|^{2k} dt &\leq k! \left(\sum_{y < p \leq x} \frac{1}{p} \right)^k + 6\pi k! (\pi(x) - \pi(y))^k / T. \end{aligned}$$

If $x \leq T^{1/k}$, the first two terms are bounded by $(Ak)^k$ and the third by $(k(\log \log x - \log \log y + A))^k$, $A > 0$ some constant.

For example, we obtain for a function $|g(u)| \leq 1$

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \left| \frac{1}{\log T^{1/V}} \sum_{n \leq T^{1/V}} \frac{\Lambda(n)}{n^{1/2+it}} g\left(\frac{\log n}{\log T^{1/V}}\right) \right|^{2[V]} dt \\ &= \frac{1}{T} \int_T^{2T} \left| \sum_{p \leq T^{1/V}} \frac{b_p}{p^{1/2+it}} + \sum_{p^2 \leq T^{1/V}} \frac{a_p}{p^{1+2it}} + O(1) \right|^{2[V]} dt \\ &\leq 3^{2V} ((AV)^V + (AV)^V + O(1)^V). \end{aligned}$$

APPENDIX C. LARGE DEVIATION THEORY

In this section we give the definition of the large deviation principle and state two important results which we used in the proofs of Corollary 1.2 and 1.3 (see [4]).

A function $I : \mathbb{R} \rightarrow [0, \infty]$ is called rate function (resp. good rate function), if for all $\alpha \in [0, \infty)$, the sets $\{x : I(x) \leq \alpha\}$ are closed (resp. compact). A family $\{Z_\epsilon\}$ of real-valued random variables satisfies the large deviation principle with speed ϵ and a rate function I , if

(a) For any closed set $F \subseteq \mathbb{R}$

$$\limsup_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}(Z_\epsilon \in F) \leq - \inf_{x \in F} I(x).$$

(b) for any open set $G \subseteq \mathbb{R}$

$$\liminf_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}(Z_\epsilon \in G) \geq - \inf_{x \in G} I(x).$$

Theorem C.1 (special case of Gärtner-Ellis' theorem, Theorem 2.3.6 or 4.5.20 in [4]). *Assume that for each $\lambda \in \mathbb{R}$*

$$\Lambda(\lambda) := \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}[e^{\lambda Z_\epsilon / \epsilon}]$$

exists and that Λ is differentiable. Then the family $\{Z_\epsilon\}$ satisfies the large deviation principle with the good rate function $I(x) = \sup_{\lambda \in \mathbb{R}} (\lambda x - \Lambda(\lambda))$.

Theorem C.2 (Varadhan, Theorem 4.3.1 in [4]). *Assume that $\{Z_\epsilon\}$ satisfies the large deviation principle with a good rate function I and let $h \in \mathbb{R}$. Assume further that for some $\gamma > 1$*

$$\limsup_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}[e^{\gamma h Z_\epsilon / \epsilon}] < \infty. \quad (\text{C.1})$$

Then

$$\lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}[e^{h Z_\epsilon / \epsilon}] = \sup_{x \in \mathbb{R}} (hx - I(x)).$$

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